Green's Theorem: Mapping Madagascar's Magnitude

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Introduction

Calculus is the mathematical study of how different factor or variable change. Calculus is used for many different things like creating models to arrive at an optimal solution, in a lot of concepts in physics like motion, harmonics, astronomy, etc and in computer soft wares like Google Earth.

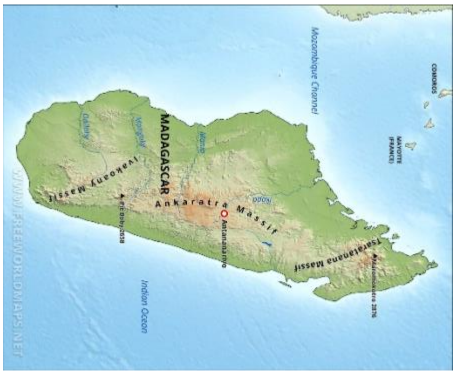
Personal engagement: When studying the IB Mathematics HL syllabus, I became privy to the concepts of calculus. I saw myself applying these concepts in other subjects like physics. I was very easily able to relate to the different concepts and applications of calculus in physics. I started using my skills of calculus of finding the gradient and area in economics as well. I wanted to see how well I could take these concepts beyond the walls of my classroom and implement them in my IA.

When I was a kid, I saw the movie Madagascar 2 and I also had the chance of visiting the beautiful island of Madagascar. I really love to travel to different countries of the world and explore each and every corner of the country. I tried to find the area of Madagascar using integration in order to know how big Madagascar is so I know how much area I would have to cover in order to travel around Madagascar. However, it is not possible to find the area of a closed curve using normal integration. So I used Green's Theorem to find the area to the closed curve. Through this paper I was able to calculate the area of Madagascar.

Rationale

In Mathematics, Green's theorem shows the relationship between the line integral of a closed curve and the double integral of the area D bounded within the closed curve. The theorem is named after George Green, who first suggested the theorem. It was first proved by Bernhard Riemann. Green's theorem is one of the theories of Calculus. In this IA, I have modelled the boundary of the Madagascar and then using Green's theorem the approximate area of the island has been found. The exploration helps check the accuracy of the area. This is to check whether I got a similar value to the actual area. Also the concept goes beyond the syllabus of IB Mathematics HL helping me to learn to calculate line integrals.

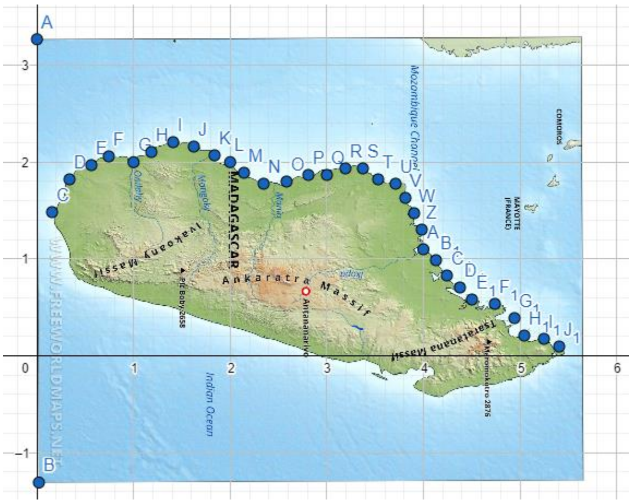
Methodology

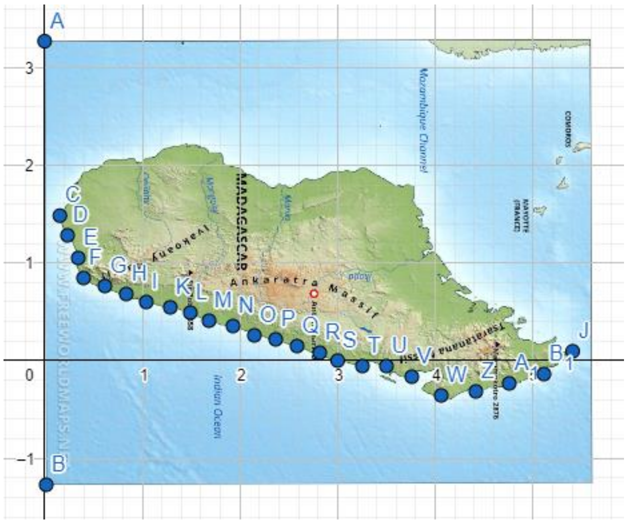


This paper is carried out using 4 steps. These steps are regression, integration, using Green's theorem and scaling. First points were plot throughout the boundary of the map of Madagascar and using these points two lines of best fit were modelled. This is called regression. Then using Integration, the area between the two curves was found. Green's theorem was then used to find the area between the closed curve, which was modelled using regression in the first step. Finally, the map was scaled to an area with appropriate units.

Regression

This is the map that was uploaded on GeoGebra, where regression was used in order to model a curve around the boundary of the map.





Upper Curve	
X_1	Y_1
0.156367	1.483868
0.337	1.822
0.5627	1.969
0.7434	2.0596
1	2
1.18137	2.1047
1.4094	2.20639
1.623	2.1612
1.8384	2.0709
2	2
2.1433	1.8902
2.3465	1.777
2.5836	1.7999
2.8093	1.8677
3.001	1.8677
3.1932	1.93544
3.3738	1.93544
3.5319	1.82255
3.71255	1.777
3.8141	1.63063
3.9044	1.4725
3.9835	1.3032
4	1.1
4.13026	0.98713
4.2431	0.82907
4.3786	0.70489
4.5028	0.5807
4.7398	0.53555
4.9431	0.3887
5.0447	0.2081
5.2479	0.1742
5.40597	0.095262

Lower Curve	
X_2	Y_2
0.156367	1.483868
0.2349	1.2825
0.34422	1.0503
0.3988	0.8454
0.6174	0.7634
0.8359	0.68152
1.04089	0.59956
1.2867	0.544922
1.4916	0.49028
1.6829	0.40831
1.9288	0.3536
2.14737	0.25805
2.3659	0.21707
2.5845	0.14877
2.8167	0.0804
3	0
3.2538	-0.0561
3.4997	-0.0561
3.7592	-0.16541
4.05981	-0.3566
4.4149	-0.3156
4.7564	-0.23371
5.1116	-0.138
5.40597	0.095262

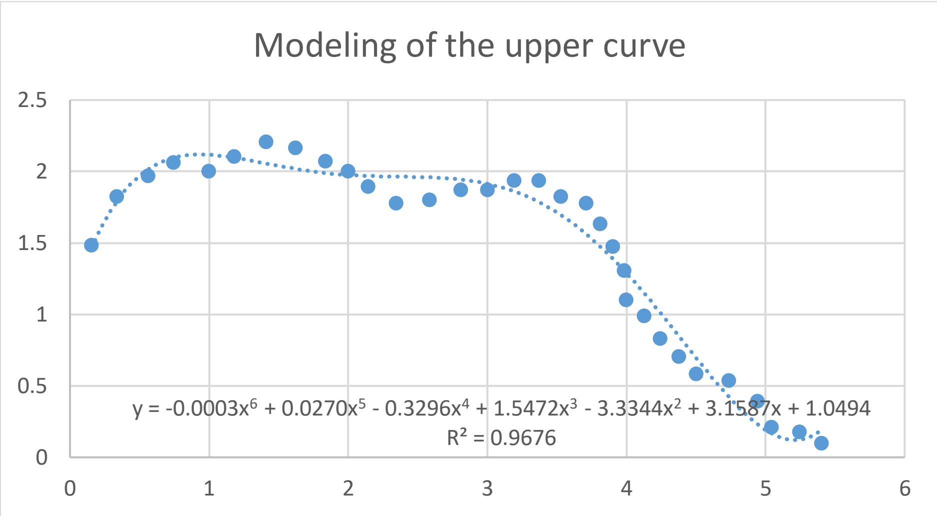
The map is in the shape of a rectangle and its dimension when put on the grid were 5.55 units x 4.52 units. When modelling the map using regression, the map was divided into 2 parts (the upper half and the lower half). Points were plotted on the boundary of the map to model the two curves. The 2 tables on the left show the points plotted to form the 2 curves. Though the original map of Madagascar is actually vertical, I had to take the map horizontally so the two modelled curves would follow the vertical line test (This is a test used to see if a curve or a function is function. This is done by visually examining how many times the vertical line intersects with the curve. If there is only one intersection, then the curve is a function). Various equations were tried in order to find the most accurate curve that model the upper and lower region's boundary.

Table 1 consists of all the points that were placed on the boundary of the upper region. 32 points were plotted on the upper region for the purpose of accuracy so the most accurate line of best fit can be modelled.

Table 2 consists of all points that were placed on the boundary of the lower region. 24 points are plotted around the lower region. This is 8 points lower than the upper region because of the topological imbalance between the upper and the lower region.

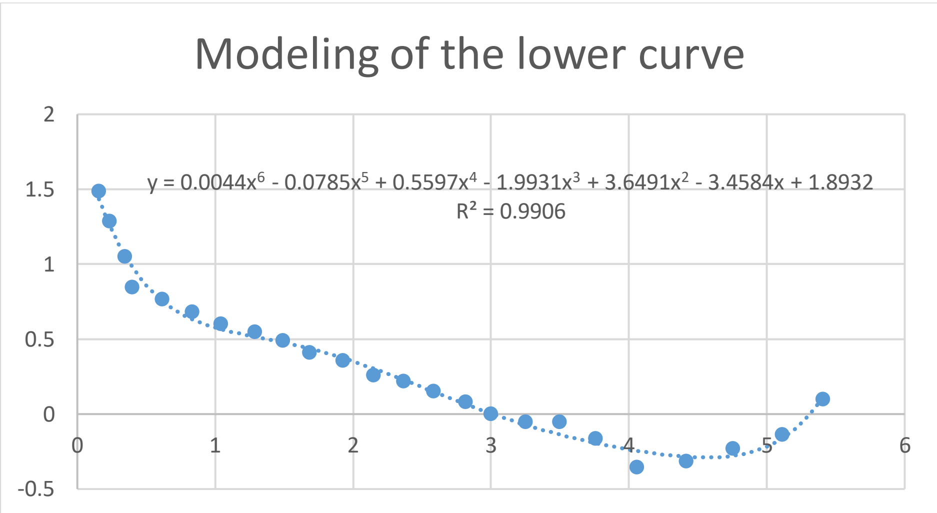
The points that were plotted in the Table 1 and Table 2 could model various number of equations with many different number of polynomials. However, the R^2 value was greatest with x^6. $R^2 = 0.96755128$ for the curve modelled by points in Table 1 and $R^2 = 0.99060068$ for the curve modelled by points in Table 2. The R^2 value denotes how accurate the line of best fit was made using the modelling points in Table 1 and Table 2. The R^2 in this case also denotes the complex topology of the upper boundary of the island making it difficult to find a line of best fit.

Figure 2 and 3 shows how the points were plotted on the boundary of Madagascar to model the two curves.



The graph above shows the line of best fit of all the points in Table 1. The equation of the curve is:

$$f(x) = -0.0003x^6 + 0.0270x^5 - 0.3296x^4 + 1.5472x^3 - 3.3344x^2 + 3.1587x + 1.0494$$



The graph above shows the line of best fit of all the points in Table 1. The equation of the curve is:

$$g(x) = 0.0044x^6 - 0.0785x^5 + 0.5597x^4 - 1.9931x^3 + 3.6491x^2 - 3.4584x + 1.8932$$

Both of the curves intersect at (0.176, 1.488) and (5.392,0.195). So a line passing through these points would divide the map into 2 parts. The equation for this line is:

$$h(x) = -0.2479x + 1.5314$$

The above figure shows the modelling of the map of Madagascar.

Integration

To find the area between *f*(x) and *g*(x) 2 main step were carried out:

Area of upper region of Madagascar

The area of the upper region of Madagascar lies *f*(x) and *h*(x) between *x* vales 0.176 to 4.821.

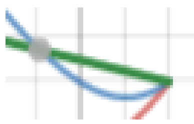
$$A_{Upper\ region} = \int_{0.176}^{4.821} f(x)\,dx - \int_{0.176}^{4.821} h(x)dx$$

$$\int_{0.176}^{4.821} f(x)\,dx = \int_{0.176}^{4.821} [-0.0003x^6 + 0.0270x^5 - 0.3296x^4 + 1.5472x^3 - 3.3344x^2 + 3.1587x + 1.0494]\,dx \approx 10.1377$$

$$\int_{0.176}^{4.821} h(x)\,dx = \int_{0.176}^{4.821} [-0.2479x + 1.5314]\,dx \approx 4.2497$$

$$\therefore A_{Upper\ region} \approx 10.1377 - 4.2497 \approx 5.888$$

Area of lower region of Madagascar



The area of the upper region of Madagascar lies *h*(x) and *g*(x) between *x* vales 0.176 to 5.392. However, there is a small area that lies between *h*(x) and *f*(x), which need to be subtracted to find the area of the lower region. The figure on the right indicates the small region.

$$\therefore A_{net\ lower\ region} = \therefore A_{lower\ region} - A_{small\ area}$$

$$\therefore A_{net\ lower\ region} = [\int_{0.176}^{5.392} h(x)\,dx - \int_{0.176}^{5.392} g(x)\,dx] - [\int_{4.821}^{5.392} h(x)\,dx - \int_{4.821}^{5.392} f(x)\,dx]$$

$$\int_{0.176}^{5.392} h(x)\,dx = \int_{0.176}^{5.392} [-0.2479x + 1.5314]\,dx \approx 4.4029$$

A part of $g(x)$ is in the 4 quadrant so to calculate $A_{lower\ region}$ we need to calculate $\int_{0.176}^{5.392} g(x)dx$

$$\int_{0.176}^{5.392} g(x)dx = \int_{0.176}^{3.354} g(x)\ dx + \int_{3.354}^{5.202} g(x)\ dx + \int_{5.202}^{5.392} g(x)\ dx$$

$$\therefore \int_{0.176}^{3.354} g(x)\,dx$$

$$= \int_{0.176}^{3.354} [0.0044x^6 - 0.0785x^5 + 0.5597x^4 - 1.9931x^3 + 3.6491x^2 - 3.4584x + 1.8932]\,dx \approx 1.3269$$

$$\therefore \int_{3.354}^{5.202} g(x)\,dx$$

$$= \int_{3.354}^{5.202} [0.0044x^6 - 0.0785x^5 + 0.5597x^4 - 1.9931x^3 + 3.6491x^2 - 3.4584x + 1.8932]\,dx \approx -0.7429$$

Because $\int_{3.354}^{5.202} g(x)\,dx$ lies in the 4 Quadrant the answer is negative. However, we want to add the area so $\int_{3.354}^{5.202} g(x)\,dx \approx 0.7429$

$$\therefore \int_{5.202}^{5.392} g(x)\,dx = \int_{5.202}^{5.392} [0.0044x^6 - 0.0785x^5 + 0.5597x^4 - 1.9931x^3 + 3.6491x^2 - 3.4584x + 1.8932]\,dx \approx 0.0723$$

$$\therefore \int_{0.176}^{5.392} g(x)dx = 1.3269 + 0.7429 + 0.0723 = 2.1421$$

$$A_{lower\ region} = 4.4029 - 2.1421 = 2.2608$$

Now I will be finding $A_{small\ area}$

$$\int_{4.821}^{5.392} h(x)\,dx = \int_{4.821}^{5.392} [-0.2479x + 1.5314]\,dx \approx 0.1531$$

$$\int_{4.821}^{5.392} f(x)\,dx = \int_{4.821}^{5.392} [-0.0003x^6 + 0.0270x^5 - 0.3296x^4 + 1.5472x^3 - 3.3344x^2 + 3.1587x + 1.0494]\,dx \approx 0.3635$$

$$A_{small\ area} = 0.1531 - 0.3635 = -0.2104$$

$$\therefore A_{net\ lower\ region} \approx 2.2608 - [-0.2104] \approx 2.5712$$

$$\therefore A_{total} \approx A_{Upper\ region} + A_{lower\ region} \approx 5.888 + 2.5712 \approx 8.4592\ square\ units$$

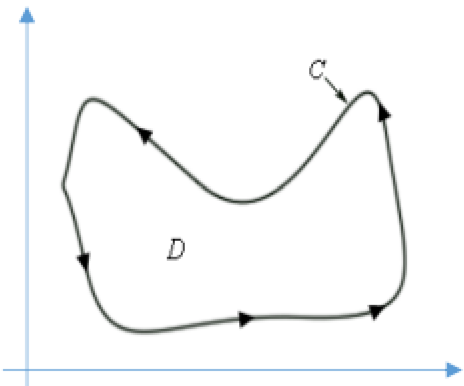
The Green's theorem

Green's theorem shows the relationship between the line integral (A line integral or path integral is the integral of some function along a curve) of a closed curve and the double integral of the area D bounded within the closed curve

The Green's theorem states that

$$\oint_c R\,dx + S\,dy = \iint_D \frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\,dx\,dy$$

In the equation C is a simple closed curve. R and S denote functions of X and Y and they have a continuous partial derivative within the region D, which is enclosed in the closed curve C. The path of the integral is always counter clockwise, which is denoted by the symbol $\oint_C$, when finding the integral of a closed curve. This symbol ∂ denotes partial derivative.



The theorem may look very intimidating; this is because two theorems are written as one:

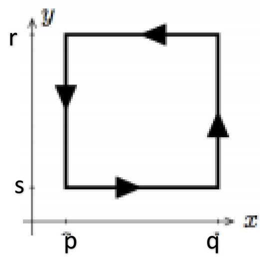
$$\oint_c S\,dy = \iint_D \frac{\partial S}{\partial x}\,dx\,dy$$

$$\oint_c R\,dx = -\iint_D \frac{\partial R}{\partial y}\,dx\,dy$$

<u>Proving Green's theorem</u>

$$\oint_c R\,dx + S\,dy = \iint_D \frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\,dx\,dy$$

$$\therefore \oint_c R\,dx = -\iint_D \frac{\partial R}{\partial y}\,dx\,dy$$



To prove Green's theorem, I have solved the right hand side of equation using the figure above.

$$\iint_D -\frac{\partial R}{\partial y}\,dy\,dx = \int_p^q \int_s^r -\frac{\partial R}{\partial y}\,dy\,dx$$

$$\int_s^r -\frac{\partial R}{\partial y}\,dy = R(x,y)_s^r = -R(x,r) + R(x,s)$$

$$\int_p^q \int_s^r -\frac{\partial R}{\partial y}\,dy\,dx = \int_p^q R(x,s) - R(x,r)\,\,dx$$

To prove Green's theorem, I have solved the left hand side of equation using the figure above.

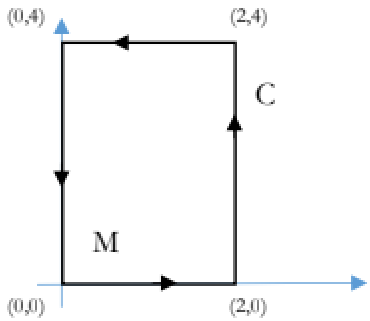
$$\oint_c R\,dx = \int_{lower\ line} R\,dx + \int_{upper\ line} R\,dx = \int_p^q R(x,s)\,dx + \int_q^p R(x,r)\,dx = \int_p^q R(x,s) - R(x,r)\,\,dx$$

Hence proved that the right hand side is equal to the left hand side.

$$LHS = RHS = \int_{p}^{q} R(x,s) - R(x,r)\ dx$$

Green's theorem example

To explain Green's theorem I will be evaluating the curve $\oint_c xy^2\ dx +\ x^5 dy$. C is a closed curve and is a rectangle. The rectangles vertices are (2,0), (2,4), (0,4), (0,0).



C follows all the criterions of Green's theorem. The orientation of C is counter clockwise and positive when plotted on the Cartesian equation above.

When comparing the $\oint_c xy^2\ dx +\ x^5 dy$ to the universal equation $\oint_c (R\ dx +\ S\ dy)$, one can say that

$$R = xy^2, S = x^5, \frac{\partial R}{\partial y} = 2xy\ and\ \frac{\partial S}{\partial x} = 5x^4$$

When substituting these values in $\iint_c \frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\ dx\ dy$

$$\int_0^4 \int_0^2 \frac{\partial S}{\partial x} - \frac{\partial R}{\partial y} dxdy = \int_0^4 \int_0^2 (5x^4 - 2xy)\ dxdy$$

$$= \int_0^4 (x^5 - x^2 y)_0^2\ dy = \int_0^4 (34 - 4y)\ dy = \ (32y - 2y^2)_0^4$$

$$= 128 - 32 = 96$$

Therefore, Green's theorem helped link a complex line integral in terms of a simple double integral.

Green's theorem manipulation to find area of Madagascar

Green's theorem is generally used to change complex line integrals into simple or basic double integrals. However, another way to go about this is to change the double integral into a line integral. This helps us easily find the area of a closed curve

$$\oint_c (R\ dx + Q\ dy) = \iint_D (\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}) dxdy$$

$$Let\ R = 0\ and\ S = x$$

$$\oint_C x\ dy = \iint_D \frac{\partial x}{\partial x} dxdy = \iint_D 1\ dxdy = A$$

$$\therefore A = \oint_C x\ dy$$

A is the area of the closed curve C and has a line integral of x dy

Similarly,

$$Let\ R = 0\ and\ S = x$$

$$\oint_C y\,dy = \iint_D -\frac{\partial y}{\partial y}dxdy = -\iint_D 1\,dxdy = A$$

$$\therefore A = -\oint_C y\,dy$$

A is the area of the closed curve C and has a negative line integral of y dy

Then, if both A expressions are added one gets:

$$\therefore 2A = \oint_C x\,dy - \oint_C y\,dy$$

$$A = \frac{1}{2}\oint_C x\,dy - \oint_C y\,dy$$

$$\boldsymbol{A = \frac{1}{2}\oint_c (x\,dy - ydx)}$$

Taking the Manipulated value to find the area

To find the area of Madagascar using Green's Theorem, the modelled equations were converted to parametric equations in terms of t.

Converting *f*(x) (Upper curve of Madagascar) to a parametric equation

$$f(x) = -0.0003x^6 + 0.0270x^5 - 0.3296x^4 + 1.5472x^3 - 3.3344x^2 + 3.1587x + 1.0494 = y_{upper\ region}$$

$$Let\ x = t\ and\ dx = dt$$

$$y_{upper\ region} = -0.0003x^6 + 0.0270x^5 - 0.3296x^4 + 1.5472x^3 - 3.3344x^2 + 3.1587x + 1.0494$$

$$\frac{dy_{upper\ region}}{dt} = -0.0019t^5 + 0.1348t^4 - 1.3182t^3 + 4.6415t^2 - 6.6689t + 3.1587$$

$$dy_{upper\ region} = [-0.0019t^5 + 0.1348t^4 - 1.3182t^3 + 4.6415t^2 - 6.6689t + 3.1587]\,dy$$

Converting *g*(x) (Lower curve of Madagascar) to a parametric equation

$$g(x) = 0.0044x^6 - 0.0785x^5 + 0.5597x^4 - 1.9931x^3 + 3.6491x^2 - 3.4584x + 1.8932$$

$$Let\ x = t\ and\ dx = dt$$

$$y_{lower\ region} = 0.0044x^6 - 0.0785x^5 + 0.5597x^4 - 1.9931x^3 + 3.6491x^2 - 3.4584x + 1.8932$$

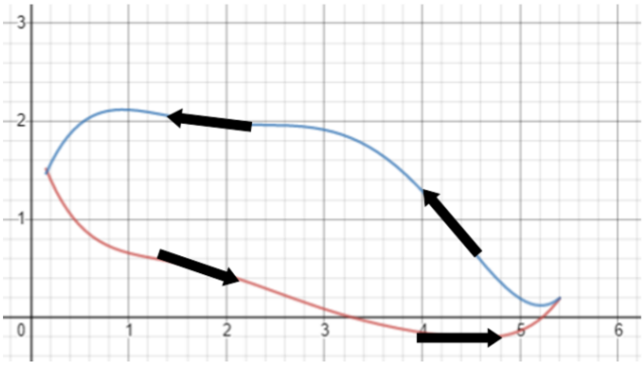
$$\frac{dy_{lower\ region}}{dt} = 0.0263t^5 - 0.3924t^4 + 2.2386t^3 - 5.9794t^2 + 7.2981t - 3.4584$$

$$dy_{lower\ region} = [0.0263t^5 - 0.3924t^4 + 2.2386t^3 - 5.9794t^2 + 7.2981t - 3.4584]dt$$

Utilising the equation above in $\boldsymbol{A = \frac{1}{2}\oint_c (x\,dy - ydx)}$

When f(x) and g(x) are converted to parametric equations, it makes it suitable to be used in

$$\boldsymbol{A = \frac{1}{2}\oint_c (x\,dy - ydx)}$$



The figure denotes f(x)(blue) and g(x) (red) and it shows the counter clockwise direction of the closed loop.

The integral of f(x) and g(x) were used according to their parametric equation (In terms of f(t) and g(t)). The integral of both the equations were added to give the area of Madagascar. The area of Madagascar gives the area between the 2 curves and it equals to the sum of the two areas A1 and A2.

$$\boldsymbol{A = \frac{1}{2}\oint_{c}(x\,dy - ydx)}$$

$$A_1 = \frac{1}{2}\int_{0.176}^{5.392} xdy_{lower\ region} - y_{lower\ region}dx =$$

$$= \frac{1}{2}\left(\int_{0.176}^{5.392} xdy_{lower\ region} - \int_{0.176}^{5.392} y_{lower\ region}dx\right)$$

$$\int_{0.176}^{5.392} xdy_{lower\ region} = \int_{0.176}^{5.392} (t)(0.0263t^5 - 0.3924t^4 + 2.2386t^3 - 5.9794t^2 + 7.2981t - 3.4584)dt = -0.5904$$

$$\int_{0.176}^{5.392} y_{lower\ region}dx = (0.0044x^6 - 0.0785x^5 + 0.5597x^4 - 1.9931x^3 + 3.6491x^2 - 3.4584x + 1.8932)dt = -382.1264$$

$$\frac{1}{2}\left(\int_{0.176}^{5.392} xdy_{lower\ region} - \int_{0.176}^{5.392} y_{lower\ region}dx\right) = \frac{1}{2}(-0.5904 + 382.1264) = 190.768$$

$$\therefore A_1 = 190.768$$

$$A_2 = \frac{1}{2}\int_{0.176}^{5.392} xdy_{upper\ region} - y_{upper\ region}dx$$

$$= \frac{1}{2}\left(\int_{0.176}^{5.392} xdy_{upper\ region} - \int_{0.176}^{5.392} y_{upper\ region}dx\right)$$

$$\int_{0.176}^{5.392} xdy_{upper\ region} = \int_{0.176}^{5.392} (t)(-0.0019t^5 + 0.1348t^4 - 1.3182t^3 + 4.6415t^2 - 6.6689t + 3.1587)dy = -45.04273$$

$$\int_{0.176}^{5.392} y_{upper\ region}\,dx$$
$$= (-0.0003x^6 + 0.0270x^5 - 0.3296x^4 + 1.5472x^3 - 3.3344x^2 + 3.1587x + 1.0494)dy = 310.7124$$

$$\frac{1}{2}\left(\int_{0.176}^{5.392} xdy_{upper\ region} - \int_{0.176}^{5.392} y_{upper\ region}\,dx\right) = \frac{1}{2}(-45.04273 - 310.7124)$$
$$= -181.7822$$

$$\therefore A_2 = -181.7822$$

$$\therefore A_{total} = A_1 + A_2 = 190.768 - 181.7822$$

$$\therefore A_{total} = 8.9858\ square\ units$$

Therefore, the value of the of the area of Madagascar from the previous section and this section is 8.2592 square units and 8.8569 square units respectively. The average of both the areas is:

$$\frac{8.9858 + 8.4592}{2} = 8.7225\ square\ units$$

In order to reduce errors and to improve accuracy an average of the 2 values is taken.

Scaling the values to find the actual area of the graph

Antanarivo (Capital of Madagascar) and Toamasina are 2 cities in Madagascar. The actual distance between the two cities is 214.85 km. The figure illustrates how the 2 points C and D are placed at Antanarivo and Toamasina respectively. C (2.74, 0.69) and D (3.32, 0.04)

$$Distance\ on\ the\ graph = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(3.32 - 2.74)^2 + (0.04 - 0.69)^2} = 0.871\ units$$

So distance between the two cities on the map is 0.871 units.

$$\therefore Actual\ area\ of\ map = \frac{214.85^2}{0.871^2} \times 8.7225 = 549363.09 \text{ square kilometres}$$

	Distance	Area
Map	0.871 units	8.7225 square units
Actual	214.85 km	549363.09 square kilometres

Limitation

Though the theorem may give an answer close to the actual value, it's not completely accurate. The actual area of Madagascar is 587,041 km^2, however we computed it to be 549363.09 km^2. These errors have occurred due to the errors that were created during regression. The points plotted during regression were very thick and large creating chances of error as the boundary did not pass through the centre of the plotted point. The R^2 value wasn't equal to 1. Those were only line of best fit and weren't in the exact shape of the boundary causing further errors. These are the different factors that caused the answer and the actual area to be different.

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